Spinal Codes over BSC: Error Probability Analysis and the Puncturing Design

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***Abstract*—As a newly invented type of rateless codes, Spinal codes can be capacity-achieving with short message length and thus hold great prospects for the design of Ultra-Reliable Low-Latency Communication (URLLC) systems. However, the error probability of Spinal codes over Binary Symmetric Channel (BSC) in the finite-length regime lacks explicit analysis in the literature, which in turn hinders efforts to the analytical design of high-efficiency associated techniques, such as the puncturing strategy. In this paper, with the bound on the number of erroneous bits in the Maximum Likelihood (ML) decoding result, we derive the asymptotically tight bound on the Bit Error Rate (BER) of Spinal codes over BSC. Based on this result, we then design the optimal puncturing strategy for Spinal codes over BSC by formulating a rate maximization problem under the constraint of low error probability. In addition, we carry out extensive simulations to verify the correctness of the error probability analysis and the effectiveness of the puncturing strategy design.**

***Index Terms*—Spinal codes, BSC, error probability analysis, puncturing.**

I. Introduction

As a newly invented type of rateless codes [1], Spinal codes can be capacity-achieving over both BSC and Additive White Gaussian Noise (AWGN) channel with short message length, which enables it to hold potentially enormous prospects for the emerging URLLC applications, including the information exchanging among self-driving cars, the user-specific 3D video rendering and augmented reality as well as the wireless automation of production facilities, etc.

Since its invention, Spinal codes have drawn extensive research interests due to its capacity-achieving property with short message length. Amount of efforts have been made for Spinal codes on its way from theory to practice. In [2], the proposed forward stack decoding (FSD) can decrease the decoding complexity remarkably without sacrificing the rate performance. In [3], the proposed sliding window decoding outperforms FSD in terms of much lower decoding complexity, while the rate performance remains almost the same. There are also lots of works on the design of Spinal codes-based protocols or architectures for mobile and wireless networking applications. In [4], the authors propose an efficiency-maximizing protocol, named RateMore, based on Spinal codes, Strider codes [5] and Raptor codes [6]. It is proved that RateMore provides a practically useful link-layer protocol in wireless networks, prominently improving the performance under the condition of time-varying channel. Moreover, in [7], the proposed cross-layer image transmission scheme significantly improves the performance of transmission efficiency with the help of Spinal codes serving as the error protector in the physical layer. In [8], the authors present a practical protocol incorporating Spinal codes, named HOPE. Experimental evaluation demonstrates that HOPE takes an advantage of network throughput over existing approaches.

The above works have made remarkable contributions on decreasing the decoding complexity and broadening the potentials of Spinal codes to future wireless communications. However, the analysis of the error probability of Spinal codes as a finite-length error control coding technique is still to be improved, especially for BSC, which is typically used as the channel model for digital transmissions. In [9], the authors put forward a method to analyze the BER over BSC by de-randomizing Shannon's random code book construction. The key idea is the application of the variant of Gallager's famous result in [10], which is unconventional since Spinal codes is not a type of traditional random code. In [11], the authors propose the average error probability upper bound on finite-length unequal error protection (UEP) Spinal codes AWGN channel and BSC, while the result is based on the average error probability applied for all kinds of random codes.

Analysis of the error probability of Spinal codes over BSC is of enormous practical value, paving the way for the analytical design of high-efficiency associated techniques, such as the concatenated coding systems [12], the forward error correction technique in the hybrid automatic repeat request (HARQ) systems [4] and the puncturing strategy design. For the optimal designs of these techniques, analysis of the error probability is a primary need.

To sum up, the analysis of the BER of Spinal codes over BSC in the finite-length regime is fundamental. However, it lacks explicit analysis in the literature. The existing analysis assumes Spinal codes as a general type of random code, ignoring its error-correction capability. The special coding structure and the rateless transmission mode of Spinal codes

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pose considerable difficulties on the analysis of the error probability of Spinal codes.

In this paper, we analyze the error probability based on the encoding structure and the decoding rule of Spinal codes rather than neglecting its error-correction capability. Firstly, we investigate the upper bound on the number of erroneous bits in the ML decoding. Then, the asymptotic BER of Spinal codes in the finite-length regime over BSC is derived. Based on the theoretical analysis of the BER, we propose the optimal puncturing strategy by formulating a rate maximization problem.

The remainder of the paper is organized as follows. Section II briefly introduces the basics of Spinal codes, including its encoding process and decoding process. The analysis of the error probability of Spinal codes over BSC is presented in section III. In Section IV, the proposed optimal puncturing strategy is presented, followed by simulation results in section V. Eventually, the conclusions are drawn in section VI.

II. Basics of Spinal Codes

A. The Encoding of Spinal Codes

Rateless Spinal codes introduce a hash function, h, as the kernel of the encoding process to continuously generate pseudo-random bits. As shown in Fig. 1, the integral encoding process includes 4 steps:

1) an n-bit message M is divided into n/k k-bit segments, denoted by m_i, where $i \in \{1, 2, \ldots, n/k\}$.
2) the encoder employs hash function to map the message segment m_i to a v-bit state s_i as

$$s_i = h(s_{i-1}, m_i), s_0 = 0^v, \tag{1}$$

where s_0 serves as the initial state known by both the encoder and the decoder.
3) the v-bit state s_i is used to seed a random number generator (RNG) to generate a sequence of pseudo-random c-bit symbols denoted by $x_{i,j}$:

$$\mathrm{RNG} : s_i \times \mathrm{N} \rightarrow x_{i,j}, \tag{2}$$

where $x_{i,j} \in \{0{,}1\}^c$, $s_i \in \{0{,}1\}^v$.
4) The sender maps the c-bit symbols to a channel input set to fit the channel characteristics: $f : x_{i,j} \rightarrow \Omega$, where f is a constellation mapping function, Ω is the channel input set.

The processes above will continue unless the decoding process succeeds and an acknowledgment (ACK) is sent to end the transmission, which reflects the rateless property of Spinal codes.

B. The Decoding of Spinal Codes

The optimal Spinal decoder is the maximum likelihood (ML) decoding. In the ML decoding, the decoder reuses the same hash function, initial spine value s_0 and RNG to completely reproduce the decoding tree. Then, the decoder traverses all the possible nodes in the decoding tree to match the maximum likely candidate sequence which is closest to

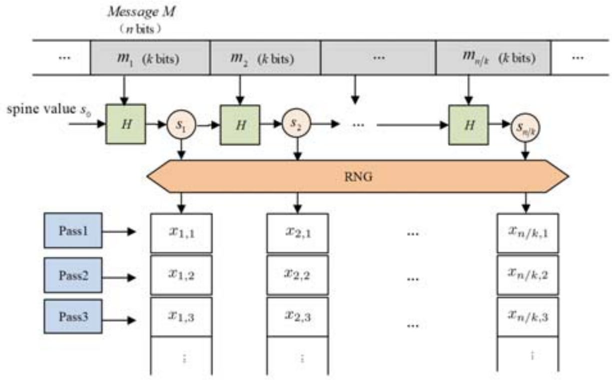

Fig. 1. The encoding process of Spinal codes

the received signals in Euclidean distance. The ML decoding rule can be expressed by

$$\hat{\mathrm{M}} = \underset{M' \in (0,1)^n}{\arg\min} \left\| \bar{y} - \bar{x}(M') \right\|^2, \tag{3}$$

where $\bar{y}$ is the vector of received symbols, $\overline{x}(M')$ represents an encoding function consisted of the same hash function, initial spine value s_0 and RNG as encoder does.

That is, the estimated message $\hat{M} \in \{0, 1\}^n$ is the one that generates a vector $\overline{x}(\hat{M})$ which is closest to $\bar{y}$ in Euclidean distance. However, traversing all the nodes in the entire decoding tree brings about an exponential increase of complexity. In [1], an ML decoder named bubble decoder is designed for Spinal codes. The bubble decoder only reserves B best matching nodes at each layer and selects the best matching one at the last layer to decode the received message.

III. Analysis of the Error Probability of Spinal Codes over BSC

The error probability analysis of Spinal codes based on the ML decoding rule over BSC can be divided into 2 steps. Firstly, we put forward the analysis of the upper bound on the number of Spinal codes' erroneous bits. Secondly, by utilizing the result obtained in the 1^{st} step, the asymptotic BER analysis of Spinal codes in the finite-length regime is elaborated.

A. Upper Bound on the Number of Erroneous Bits

In [1], J. Perry indicates that there exists an upper bound on the path cost during the decoding process of Spinal codes. To elaborate it, we assume $\alpha\bar{y}$ as the vector of received symbols, where α is the correction factor of linear minimum mean square error (LMMSE). By introducing the parameter α, the ML decoder can be approximated as a LMMSE decoder [1]. The correction factor can be calculated by $\alpha = P^* / \left(P^* + \sigma^2\right)$, where P^* denotes the average power of transmitted signal, σ^2 denotes the variance of the noise.

Theorem 1. *For any* $1 \leq i \leq L$ *over AWGN channel, and any* $\varepsilon > 0$*, we possess probability of* $1 - O\left(\exp\left(-\Theta\left(\varepsilon^2 iL\right)\right)\right)$ *to assure that*

$$\sum_{j=1}^{i}\sum_{l=1}^{L}\left(\alpha y_{j,l} - x_{j,l}(M)\right)^2 \leq (1+\varepsilon)\frac{iLP^*}{1+SNR}. \quad (4)$$

Proof. Consider a general transmission over AWGN channel as $y_{j,l} = x_{j,l}(M) + N_{j,l}$, where $N_{j,l}$ denotes the independent Gaussian noise $\sim \left(0, \sigma^2\right)$, we obtain

$$\begin{aligned}\left(\alpha y_{j,l} - x_{j,l}(M)\right)^2 =& \alpha^2 N_{j,l}^2 + (1-\alpha)^2 x_{j,l}(M)^2 \\ & - 2\alpha(1-\alpha) n_{j,l} x_{j,l}(M).\end{aligned} \quad (5)$$

Note that $x_{j,l}(M)$ and $N_{j,l}$ are independent:

$$\begin{aligned}E\left(\left(\alpha y_{j,l} - x_{j,l}(M)\right)^2\right) &= \alpha^2\sigma^2 + (1-\alpha)^2 P^* \\ &= \frac{P^*\sigma^2}{(P^*+\sigma^2)} = \frac{P^*}{1+SNR}\end{aligned}. \quad (6)$$

Then, for small enough $\varepsilon > 0$, applying the Chernoff bound, we can easily obtain (4). □

Define Spinal codes with message length n, segment length k and pass number L as (n, k, L) Spinal codes. For BSC with crossover probability f, the corresponding Signal Noise Ratio (SNR) is equivalent to $\left[\mathrm{erfc}^{-1}(2f)\right]^2$. Meanwhile, for BSC, the Euclidean distance can be replaced by Hamming distance such that

$$\begin{aligned}\sum_{l=1}^{L}\left(\alpha y_{j,l} \oplus x_{j,l}(M)\right) &= \sum_{l=1}^{L}\left(\alpha y_{j,l} - x_{j,l}(M)\right)^2 \\ &\leq (1+\varepsilon)\frac{LP^*}{1+\left[\mathrm{erfc}^{-1}(2f)\right]^2}.\end{aligned} \quad (7)$$

Remark: For BSC, the constellation mapping is trivial: $c = 1$, which means that the one-bit symbol $x_{j,l}(M)$ is directly transmitted to the channel without constellation mapping.

It turns out that the left hand side of (7) represents the number of erroneous bits for each encoded segment. Considering that the number of erroneous bits must be an integer, we set Q as

$$Q = \left\lfloor (1+\varepsilon)\frac{LP^*}{1+\left[\mathrm{erfc}^{-1}(2f)\right]^2}\right\rfloor. \quad (8)$$

to denote the maximum number of erroneous bits for one segment of Spinal codes.

B. Asymptotic Performance Analysis of Spinal Codes over BSC

We denote *m* as the transmitted sequence of one segment, the probability of the error event E is

$$\mathbb{P}(E) \triangleq \sum_{m}\mathbb{P}(E|m)\,\mathbb{P}(m). \quad (9)$$

Assume *m* to be equiprobable. Due to symmetry of the random code, $\mathbb{P}(E)$ is equal to $\mathbb{P}(E|m)$ for any given *m*. Let $\hat{m}$ denote the estimated sequence of the received message generated by the ML decoder, it turns out that

$$\mathbb{P}(E|m) \triangleq \mathbb{P}(\hat{m} \neq m|m) \quad (10)$$

Therefore, for the (n, k, L) Spinal code, the error probability of each segment can be upper bounded as

$$\mathbb{P}(E) = \mathbb{P}(\hat{m} \neq m|m) \leq \sum_{i=1}^{Q}\binom{L}{i} f^i (1-f)^{L-i}. \quad (11)$$

Considering that (11) is based on the idea of permutation and combination, it somehow over-evaluates the upper bound on error probability without considering the error-correction capability of Spinal codes. According to Spinal codes' property, (11) can be modified by treating Spinal codes as a kind of special linear error-correcting code.

We introduce S_w, the number of the sequences produced by the spine value with hamming weight w, to modify (11). Let d denote the minimum distance and $t_w = \lfloor w/2 \rfloor$ represent the lower bound on the number of erroneous bits for each encoded segment [13]. Then for the (n, k, L) Spinal codes, the asymptotic tight bound on the bit error probability is

$$P_e \leq \left[\sum_{w=d}^{L} S_w \sum_{i=t_w}^{Q}\binom{L}{i} f^i (1-f)^{L-i}\right] / (nL/k) \quad (12)$$

where $\{S_w\}$, w, and d are obtained by statistic data.

IV. Optimal Puncturing Design for Spinal Codes over BSC

During the transmission process of Spinal codes, generally a whole pass is transmitted to ensure the decoding requirements. However, there always exists redundant symbols in the pass, which results in a loss of coding rate. The puncturing strategy is introduced to reduce the number of symbols transmitted through each pass and improve the transmission rate. In [1], the authors propose the uniform puncturing design, however, the design ignores the serial coding structure of Spinal codes, which can be enhanced by the proposed optimal puncturing design elaborated in this section.

A. The Optimization Problem Formulation

To establish the mathematical model, two fundamental parameters are emphasized here, including coding rate and Frame Error Rate (FER):

1)$Coding\ Rate$: The puncturing strategy separates the entire pass into individual sub-passes to transmit the message. Let $l_i, i \in \{1, 2, \ldots, n/k\}$ denote the number of passes transmitted by the i^{th} segment. The coding rate can be expressed as

$$R = n / \sum_{i=1}^{n/k} l_i. \quad (13)$$

2)$Frame\ Error\ Rate$: Eq.(12) proposes the bound on the bit error probability, however, it should be modified a little to adapt to the situation where L is not fixed.

From (8), the upper bound on the erroneous bits of the i^{th} segment is

$$Q_i = \left\lfloor (1+\varepsilon) \frac{L_i P^*}{1 + \left[\mathrm{erfc}^{-1}(2f)\right]^2} \right\rfloor, \tag{14}$$

where $L_i = \left\{ \sum_{m=n/k+1-i}^{n/k} l_m, i = 1, 2, \ldots, n/k \right\}$.

Similar to (12), the BER bound on the the i^{th} segment is

$$P_{e_i} \leq \left[\sum_{w_i = d_i}^{L_i} S_w \sum_{j=t_{w_i}}^{Q_i} \binom{L_i}{j} f^j (1-f)^{L_i - j} \right] / L_i, \tag{15}$$

where w_i is the hamming weight of the i^{th} segment, d_i is the minimum hamming distance of the i^{th} segment, $t_{w_i} = \lfloor w_i/2 \rfloor$.

Let P_{ef} denote FER and E_i denote the event that there exists an error in the i^{th} segment. Then, it turns out that

$$\begin{aligned} P_{ef} &= \mathbb{P}(E_1 \cup E_2 \cup \ldots \cup E_{n/k}) \\ &= 1 - \prod_{i=1}^{n/k} \mathbb{P}(\bar{E}_i | \bar{E}_1 \ldots \bar{E}_{i-1}) \\ &\leq 1 - \prod_{i=1}^{n/k} (1 - P_{e_i}), \end{aligned} \tag{16}$$

where P_{e_i} is upper bounded by (12)

Then, the mathematical model is established as below:

1) Constraint: To ensure reliable communication, the Spinal decoding error probability should be smaller than a preset threshold of URLLC system [14]:$P_{ef}^{\text{upper}} \leq 10^{-5}$.
2) Variable: $l_i, i \in \{1, 2, \ldots, n/k\}$, the number of passes transmitted by the i^{th} segment.
3) Objective function: The maximum code rate, denoted as

$$R_{max} = n / \sum_{j=1}^{n/k} l_j \tag{17}$$

Summarize the above into mathematical expressions, we obtain:

$$\left\{ \hat{l}_j \right\} = \arg\max_{\{l_j\}} R$$

$$\text{s.t.} \quad \begin{cases} P_{ef}^{\text{upper}} = 1 - \prod_{i=1}^{n/k} (1 - P_{e_i}) \leq 10^{-5} \\ 0 < l_1 \leq l_2 \leq \cdots \leq l_{n/k} \\ l_1, l_2, \ldots, l_{n/k} \in \mathbb{Z}^+ \end{cases}$$

This model is a typical nonlinear integer programming (NLP) problem, which can be figured by optimization algorithm tools. Some results are given in the next section.

B. Case Study

Take $n = 32, k = 4$ as an example, the value of l_i is displayed in Tab. I.

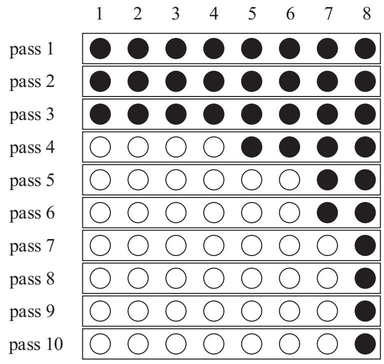

Fig. 2. The optimal puncturing strategy under p=0.01. In each pass, the sender only transmits the symbols marked by dark circles and the symbols marked by white circles will be punctured.

TABLE I
THE OPTIMAL PUNCTURING STRATEGY UNDER EACH p

p	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8
0.1	3	4	4	4	5	5	8	11
0.08	3	3	4	4	4	5	7	11
0.05	3	3	3	4	4	4	7	10
0.01	3	3	3	3	4	4	6	10
0.005	3	3	3	3	3	4	5	8
0.001	3	3	3	3	3	3	4	6
0.0001	3	3	3	3	3	3	4	4

In the decoding process of Spinal codes, the child node is extended in the right way only when the parent node is decoded correctly. In this way, we can conclude that the tail symbols of Spinal codes are more likely to be erroneous. By observing the puncturing patterns in Tab. I, we can find that the proposed puncturing strategy tends to transmit more tail symbols, which provides better protection of error-prone tail symbols and reduces the transmission redundancy.

V. SIMULATION RESULTS

In this section, simulation results are given to verify the accuracy of the asymptotic analysis and demonstrate the superiority of the proposed optimal puncturing strategy.

For comparison, Fig. 3 shows the BER performance of simulation results and asymptotic analysis. The simulation is carried out under the parameter setting as $L = 12$, $n = 32$, $k = 4$, $B = 64$. The gap between the simulation curve and the upper bound on BER increases as the crossover probability descends. This phenomenon can be explained as follows: as the crossover probability decreases, the number of erroneous bits upper bound Q gradually declines, posing difficulty on carrying out accurate asymptotic analysis.

Fig. 4 shows the rate performance comparison among the optimal puncturing strategy and the original uniform puncturing strategy as well as Spinal codes without puncturing over BSC. The simulation results of the rate performance are obtained in the mode of rateless transmission and the transmission parameters are set as $n = 32$, $k = 4$, $B = 64$. Obviously, the proposed puncturing strategy achieves better rate performance than the uniform puncturing strategy.

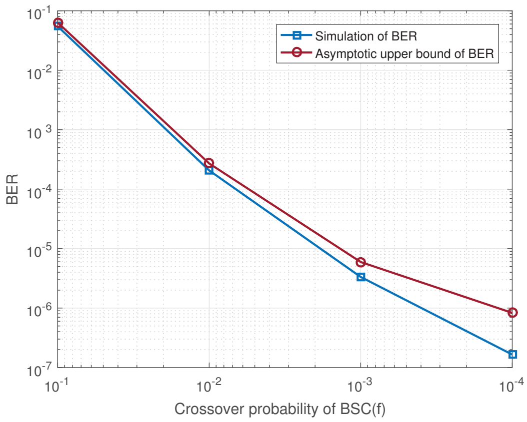

Fig. 3. The comparison of BER performance between simulation and the asymptotic analysis

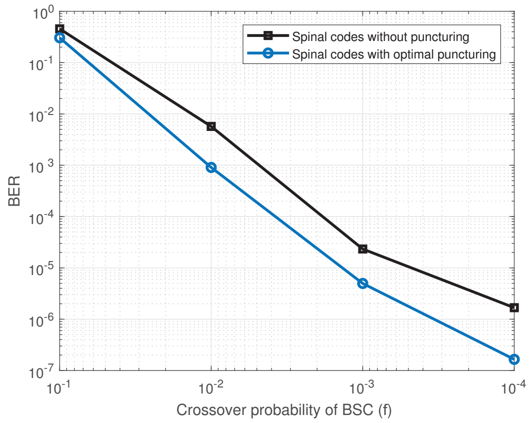

Fig. 5. The BER performance of the optimal puncturing strategy and the original Spinal codes.

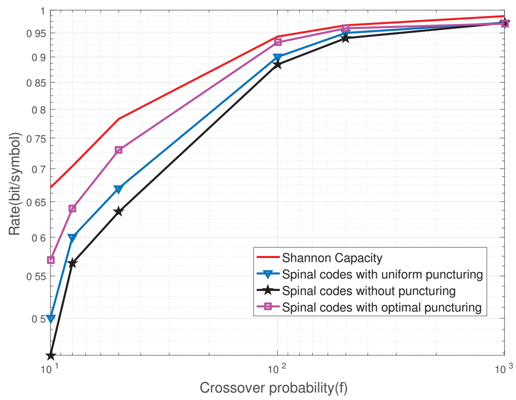

Fig. 4. The rate performance of the optimal puncturing strategy

In Fig. 5, the BER performance comparison between the optimal puncturing strategy and the original Spinal codes is presented, where the Spinal code is utilized as a fixed-rate code. To follow the principle of single variable, the transmission parameters are set as $n = 32$, $k = 4$, $B = 64$, with the fixed rate set as $R = k/L = 1/3$ bit/symbol. The simulation results illustrate that the optimal puncturing strategy outperforms the original Spinal codes. The optimal puncturing strategy's advantage of BER performance can be attributed to its tendency of tail-symbols transmitting. That is, compared to the original Spinal codes under the premise of the same rate, the proposed puncturing strategy sends more symbols for the error-prone tail parts to ensure successful decoding, which in turn reduces the bit error rate and leads to a better BER performance.

VI. Conclusion

Analysis of the error probability of Spinal codes over BSC is of fundamental importance. It serves as the theoretical basis for the analytical design of Spinal codes-based high-efficiency associated techniques in URLLC systems. With the upper bound on the number of erroneous bits in the ML decoding result, we derive the BER upper bound of Spinal codes in the finite-length regime over BSC. The simulation shows that the error probability analysis well approximates the BER performance of Spinal codes. Based on the theoretical analysis, we also propose the optimal puncturing strategy for Spinal codes, which outperforms the uniform puncturing strategy and achieves higher rate in the simulation.

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